MATH1003

ASSIGNMENT 8
ANSWERS

1. Be careful! Since $1 - \sin\theta \to 0$ and $\csc\theta \to 1$ as $\theta \to \pi/2$, L'Hôpital's Rule does not apply. But this isn't a problem, since we need only use the Laws of Limits:

$$\begin{aligned}\lim_{\theta\to\pi/2} \frac{1-\sin\theta}{\csc\theta} &= \frac{0}{1}\\ &= 0.\end{aligned}$$

2. Proposition. *For any $\rho > 0$,*

$$\lim_{x\to\infty} \frac{\ln x}{x^\rho} = 0$$

Proof. Observe that both $\ln x \to \infty$ and $x^\rho \to \infty$ as $x \to \infty$. Hence we can apply L'Hôpital's Rule to obtain that:

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln x}{x^\rho} &= \lim_{x\to\infty} \frac{1/x}{\rho x^{\rho-1}}\\ &= \lim_{x\to\infty} \frac{1}{\rho x^\rho}\\ &= 0.\end{aligned}$$

□

3. (i) Let $f(x) = x^3 + x^2 - x$. Then:

$$\begin{aligned}\frac{dy}{dx} &= 3x^2 + 2x - 1\\ &= (3x-1)(x+1),\end{aligned}$$

which is defined everywhere, and equals zero when $x = -1$ or $x = 1/3$. Hence the critical numbers are $x = -1$ and $x = 1/3$.

(ii) Let $g(\theta) = 4\theta - \tan\theta$. This is undefined when $\theta = (2k+1)\pi/2$, where $k \in \mathbb{Z}$. The derivative is given by:

$$g'(\theta) = 4 - \sec^2\theta.$$

http://erdos.math.unb.ca/~kasprzyk/
kasprzyk@unb.ca.

This is defined whenever g is defined, and is zero when:

$$
\begin{aligned}
& \sec^2\theta = 4 \\
\Rightarrow\quad & \cos^2\theta = \frac{1}{4} \\
\Rightarrow\quad & \cos\theta = \pm\frac{1}{2} \\
\Rightarrow\quad & \theta = 2k\pi \pm \frac{\pi}{3}, (2k+1)\pi \pm \frac{\pi}{3}, \qquad \text{where } k \in \mathbb{Z} \\
\Rightarrow\quad & \theta = k\pi \pm \frac{\pi}{3}.
\end{aligned}
$$

Hence the critical numbers are at $k\pi \pm \pi/3$, for all $k \in \mathbb{Z}$.

4. (i) First note that f is continuous on this interval. We have that:

$$
\begin{aligned}
f'(x) &= 3x^2 - 12x + 9 \\
&= 3(x-1)(x-3).
\end{aligned}
$$

This is defined for all x in $(-1, 4)$, and is zero when $x = 3$ or $x = 1$. Hence the critical values are $x = 3$ and $x = 1$. The Closed Interval Method tells us that the global minimum and global maximum values of f on $[-1, 4]$ are given by one of the critical values or by the end points.

x	$f(x)$
-1	-14
1	6
3	2
4	6

Hence the global maximum value is given by 6, and occurs when $x = 1$ and when $x = 4$. The global minimum value is -14, and occurs when $x = -1$.

(ii) Differentiating f gives:

$$
\begin{aligned}
f'(x) &= 6x^2 - 6x - 12 \\
&= 6(x^2 - x - 2) \\
&= 6(x+1)(x-2).
\end{aligned}
$$

This is zero when $x = -1$ and when $x = 2$; these are the critical values. The global maximum and global minimum values of f occur at the critical values, or at the end points of the domain. Hence we need to calculate:

x	$f(x)$
-2	-3
-1	8
2	-19
3	-8

The global maximum value is 8 (when $x = -1$), and the global minimum value is -19 (when $x = 2$).

(iii) Differentiating f gives:

$$\begin{aligned} f'(x) &= 3 \times 2x \times (x^2 - 1)^2 \\ &= 6x(x^2 - 1)^2. \end{aligned}$$

This is zero when $x = 0$ and when $x = \pm 1$; these are the critical values. The global maximum and global minimum values of f occur at the critical values, or at the end points of the domain. Hence we need to calculate:

x	$f(x)$
-1	0
0	1
1	0
2	9

The global maximum value is 9 (when $x = 2$), and the global minimum value is 0 (when $x = -1$ and when $x = 1$).

(iv) Differentiating f gives:

$$\begin{aligned} f'(x) &= e^{-x} - xe^{-x} \\ &= (1 - x)e^{-x}. \end{aligned}$$

This is zero when $x = 1$; this is the only critical value. The global maximum and global minimum values of f occur at a critical value, or at the end points of the domain. Hence we need to calculate:

x	$f(x)$
0	0
1	e^{-1}
2	$2e^{-2}$

The global minimum value is 0 (when $x = 0$). Observe that:

$$
\begin{aligned}
& e^{-1} < 2e^{-2} \\
\Rightarrow \qquad & e < 2
\end{aligned}
$$

which is a contradiction. Hence $e^{-1} > 2e^{-2}$ and so the global maximum value is e^{-1} (when $x = 1$).