MATH1003

ASSIGNMENT 9
ANSWERS

1. Let $f : [0,2] \to \mathbb{R}$ be given by $f(x) = x^3 + x - 1$. This is continuous on the given interval and, since f is a polynomial, differentiable on $(0,2)$. Hence it satisfies the conditions of the Mean Value Theorem.

The Mean Value Theorem tells us that there exists some $c \in (0,2)$ such that:

$$\begin{aligned} f'(c) &= \frac{f(2) - f(0)}{2 - 0} \\ &= \frac{9 - (-1)}{2} \\ &= 5. \end{aligned}$$

Since $f'(x) = 3x^2 + 1$, we see that $3c^2 + 1 = 5$ and so $c = 2/\sqrt{3}$.

2. Proposition. *Let f and g be continuous on $[a,b]$ and differentiable on (a,b). Suppose also that $f(a) = g(a)$ and $f'(x) < g'(x)$ for $a < x < b$. Then $f(b) < g(b)$.*

Proof. Let $h = f - g$. Then, since both f and g are continuous on $[a,b]$, h is continuous on $[a,b]$. Because f and g are differentiable on (a,b), so h is differentiable on (a,b). Hence h satisfied the conditions on the Mean Value Theorem, and there exists some $c \in (a,b)$ such that:

$$\begin{aligned} h'(c) &= \frac{h(b) - h(a)}{b - a} \\ \Rightarrow \qquad f'(c) - g'(c) &= \frac{f(b) - g(b) - f(a) + g(a)}{b - a}. \end{aligned}$$

By hypothesis $f'(c) < g'(c)$, and so $f'(c) - g'(c) < 0$. Since $f(a) = g(a)$ we obtain that:

$$\begin{aligned} \frac{f(b) - g(b)}{b - a} &< 0 \\ \Rightarrow \qquad f(b) - g(b) &< 0 \\ \Rightarrow \qquad f(b) &< g(b), \end{aligned}$$

as required. ∎

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3. (i) f is increasing when the derivative f' is positive. This occurs in the intervals $(2, 4)$ and $(6, 9]$.

(ii) f has a local minimum when f' changes from negative sign to positive sign. This occurs at $x = 2$ and $x = 6$. There is a local maximum when f' changes from positive sign to negative sign. This occurs at $x = 4$. We must also consider the endpoints $x = 0$ and $x = 9$. Close to $x = 0$ the gradient is negative, hence we have a local maximum there. Close to $x = 9$ the gradient is positive, and we see that this is also a local maximum.

(iii) By the Concavity Test, f is concave upwards in the regions where f'' is positive. Thus we require f' to have a positive gradient. This occurs in the intervals $(1, 3)$, $(5, 7)$ and $(8, 9]$. f is concave downwards when f'' is negative. We thus require f' to have negative gradient. This occurs in the intervals $[0, 1)$, $(3, 5)$ and $(7, 8)$.

(iv) By definition a point of inflection is when f swaps from being concave upwards to concave downwards, or vice versa. This occurs when $x = 1$, $x = 3$, $x = 5$, $x = 7$, and $x = 8$.

4. Let $B(x) = 3x^{2/3} - x$. Then $B'(x) = 2x^{-1/3} - 1$ and $B''(x) = -(2/3)x^{-4/3}$.

(i) B is increasing when B' is positive. This occurs when $2x^{-1/3} - 1 > 0$; i.e. when $2 > \sqrt[3]{x}$. Hence when $x < 8$. B is decreasing when B' is negative. We see that this occurs when $x > 8$.

(ii) Local maxima or minima occur when B' changes sign, or at the boundary of the domain on which the function is defined. B' changes sign once, when $x = 8$, from positive to negative. This implies that B has a local maximum at the point $(8, 4)$. Since B is defined only on $[0, \infty)$, we need to consider the point $x = 0$. Close to $x = 0$ B has positive gradient. Hence there is a local minimum at the origin $(0, 0)$.

(iii) By the Concavity Test, B is concave upwards in the regions where B'' is positive, and is concave downwards when B'' is negative. Hence B is concave upwards when $-(2/3)x^{-4/3} > 0$, but since $x \geq 0$ this never occurs. On the other hand, $-(2/3)x^{-4/3} < 0$ for all $x \geq 0$, and so B is concave downwards for all points in its domain.

By definition a point of inflection is when B swaps from being concave upwards to concave downwards, or vice versa. Since B is never concave upwards, there can be no points of inflection.

(iv) Sketching the graph is easy.

5. Consider $y = (x^2 - 2)/x^4$. This has domain $\mathbb{R} \setminus \{0\}$. Since:

$$\frac{(-x)^2 - 2}{(-x)^4} = \frac{x^2 - 2}{x^4},$$

the function is even. Hence the graph is symmetric about the y-axis.

(i) The roots $y = 0$ occur when $x^2 - 2 = 0$; i.e. when $x = \pm\sqrt{2}$.

(ii) We consider the behaviour of the graph close to $x = 0$:

$$\lim_{x \to 0^+} \frac{x^2 - 2}{x^4} = -\infty$$

By symmetry,

$$\lim_{x \to 0^-} \frac{x^2 - 2}{x^4} = -\infty.$$

Hence we have a vertical asymptote at $x = 0$.

(iii) For large values of x,

$$\lim_{x \to \infty} \frac{x^2 - 2}{x^4} = 0.$$

By symmetry,

$$\lim_{x \to -\infty} \frac{x^2 - 2}{x^4} = 0.$$

(iv) Now we consider the derivatives.

$$\begin{aligned} \frac{dy}{dx} &= \frac{2x^5 - 4x^3(x^2 - 2)}{x^8} \\ &= 2\left(\frac{x^2 - 2(x^2 - 2)}{x^5}\right) \\ &= 2\left(\frac{4 - x^2}{x^5}\right). \end{aligned}$$

Hence $dy/dx = 0$ when $x = \pm 2$. Since dy/dx is defined for all x in the domain of our function, these are the only critical points.

$$\begin{aligned} \frac{d^2y}{dx^2} &= 2\left(\frac{-2x^6 - 5x^4(4 - x^2)}{x^{10}}\right) \\ &= 2\left(\frac{-2x^2 - 5(4 - x^2)}{x^6}\right) \\ &= 2\left(\frac{3x^2 - 20}{x^6}\right). \end{aligned}$$

Since $x^6 > 0$ for all x in the domain, the sign of d^2y/dx^2 depends solely on the sign of $3x^2 - 20$. Thus:

$$\frac{d^2y}{dx^2} \begin{cases} = 0, & \text{if } x = \pm 2\sqrt{5/3}; \\ < 0, & \text{if } |x| < 2\sqrt{5/3}; \\ > 0, & \text{otherwise.} \end{cases}$$

Hence, since $\sqrt{5/3} > 1$, we see that $d^2y/dx^2 < 0$ when $x = \pm 2$. Hence we have a local maximum at $x = \pm 2$. Since the sign of d^2y/dx^2 changes at $x = \pm 2\sqrt{5/3}$ we see that the graph has a point of inflection at these points.

(v) This is enough information with which to sketch the graph.