MATH1003
REVISION

1. Differentiate the following functions, simplifying your answers when appropriate:

(i) $f(x) = (x^3 - 2)\tan x$

(ii) $y = (3x^5 - 1)^6$

(iii) $y^2 = x^2 - 3$

(iv) $y = \ln(\ln(7 + x)) - e^{5x^3}$

(v) $g(t) = \dfrac{\cos t}{2\sin 2t}$

(vi) $y = \cosh\sqrt{t} - \sinh\sqrt{t}$

(vii) $y \sin x^2 = x \sin y^2$

(viii) $xy = \cot(xy)$

(ix) $1 + x = \sin(xy^2)$

(x) $g(x) = \dfrac{1 - \cosh x}{1 + \cosh x}$

2. (i) Differentiate the following functions:

(a) $y = 2^{x^3}$

(b) $y = 3^{4^{x^5}}$

(c) $2x^y = 1$

(d) $x^{2y} - 5 = 0$

(ii) By taking e of both sides, show that $\ln a = b$ if and only if $e^b = a$.

(iii) More generally we have the following:

$$\log_a b = c \qquad \text{if and only if} \qquad a^c = b.$$

By using this relation, differentiate:

$$y = \log_x 2.$$

http://erdos.math.unb.ca/~kasprzyk/
kasprzyk@unb.ca.

3. (i) Use logarithmic differentiation to find $\frac{dy}{dx}$ for the following functions. Be sure to simplify your answers:

(a) $y = (5x^2+2)^4(x+7)^3$

(b) $y = (2x-7)^3(x^3+1)^9$

(c) $y = (7x^2-2)^4(x+7)^3$

(ii) By using logarithmic differentiation, or otherwise, show that the following functions have the given derivative:

(a) $y = \dfrac{(3x^2+1)^2}{\sqrt{9x^4-1}}$ and $\dfrac{dy}{dx} = \dfrac{6x(3x^2-2)(3x^2+1)^2}{(9x^4-1)^{3/2}}$

(b) $y = \dfrac{(x^2+1)^5}{\sqrt{x^4-1}}$ and $\dfrac{dy}{dx} = \dfrac{2x(6x^2-5)(x^2+1)^5}{(x^4-1)^{3/2}}$

(c) $y = \dfrac{(2x+1)^7}{\sqrt{4x^2-1}}$ and $\dfrac{dy}{dx} = \dfrac{2(5x-1)(2x+1)^7}{(4x^2-1)^{3/2}}$

4. (i) Differentiate $y = \cos rx$ twice, where r is a constant, to show that:

$$y'' + r^2 y = 0.$$

Hence write down a solution to $y'' + 25y = 50$.

(ii) Differentiate $y = \cosh(ax) + \sinh(ax)$ twice, where a is a constant, to show that:

$$y'' - a^2 y = 0.$$

Hence write down a solution to $y'' - 9y = 27$.

(iii) Differentiate $y = e^{rx^2}$ twice, where r is a constant, to show that:

$$y'' - 2rxy' - 2ry = 0.$$

Hence write down a solution to $y'' - xy' - y = 0$.

5. (i) Find an expression for $\frac{dy}{dx}$ for the following curves:

(a) $3x^2 - 2y^2 = 1$

(b) $y = x \ln x$

(c) $y^4 = 2x^2 - 1$.

(ii) For each case in (i) prove that the tangent to the curve is *never* parallel to the x-axis.

6. Find the equation of the tangent line to $y = \ln 3x$ at the point where the tangent line is parallel to the line $2y - 6x = 1$.

7. Consider the curve given by $x^2 + y^2 = x + y + 4$.

(i) Find $\dfrac{dy}{dx}$ for this curve.

(ii) Verify that $(-1, 2)$ lies on the curve.

(iii) Find the equation of the tangent line to the curve at $(-1, 2)$.

8. Consider the curve $x^2 + 2xy + 10y^2 = 9$.

(i) Find an expression for $\dfrac{dy}{dx}$.

(ii) Verify that $(3, 0)$ is a point on the curve.

(iii) Find the equation of the line tangent to the curve at the point $(3, 0)$.

(iv) Find both points on the curve where the tangent line is horizontal.

9. Consider the curve $3x^2 - xy + 2y^2 = 12$.

(i) Find the values of y when $x = 1$.

(ii) Find an expression for $\dfrac{dy}{dx}$.

(iii) For each point found in (i), calculate the equation of the tangent line.

10. Consider the curve $x^5 + xy + y^7 = 1$. Find the slope of the tangent line at the point P where this curve crosses the y-axis.

11. (i) Using the fact that

$$(\cosh t)^2 - (\sinh t)^2 = 1$$

prove that:

$$(\operatorname{sech} t)^2 = 1 - (\tanh t)^2.$$

(ii) Let $y = \tanh^{-1} x$ be the inverse of $\tanh x$. Show that:

$$\frac{dy}{dx} = \frac{1}{1 - x^2}.$$

(iii) Let $f(x) = \tanh^{-1}(\sin x)$. Prove that:

$$f'(x) = \sec x.$$

12. (i) Sketch $y = \dfrac{x}{1-x^2}$ between $-1 < x < 1$. On a separate graph sketch $y = \tan x$ between $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

(ii) Sketch $y = \dfrac{x-1}{x(x-3)}$ between $0 < x < 3$ and $y = \cot x$ between $0 < x < \pi$.

(iii) Sketch $y = x^2+1$. Which hyperbolic trigonometric function does this remind you of? Sketch it.

(iv) Sketch $y = \dfrac{2x}{\sqrt{9x^2+4}}$. Find a similar–looking inverse trigonometric function.

13. Find an inverse for the following functions. In each case be careful to state the domain of the inverse function.

(i) $y = x^2 - 1$

(ii) $y = \cos 3x$

(iii) $y = \sin^{-1}(x+5)$

(iv) $y = \tan^{-1}\sqrt{x}$

14. Find explicit formulas for the following functions:

(i) $y = \cosh^{-1} x$

(ii) $y = \cosh^{-1}(x+1)$

(iii) $y = \sinh^{-1} x$

(iv) $y = \sinh^{-1}\sqrt{x}$

15. Calculate the asymptotes, then sketch the following functions:

(i) $y = \dfrac{3x^2-2}{x^2-1}$

(ii) $y = \dfrac{x^2}{x^2-3x-10}$

(iii) $y = \dfrac{x^2+3x-1}{x^2-1}$

(iv) $y = \ln x$

(v) $y = \dfrac{x}{\sqrt{2-x^2}}$

(vi) $y = \dfrac{3-2x^2}{x^2+2x}$

16. Evaluate the following limits. If you make use of L'Hospital's rule, make sure that you justify its usage.

(i) $\lim_{x\to\infty} \dfrac{x^2 - 4x^3}{1 + x^3}$

(ii) $\lim_{x\to 0} \dfrac{\tan 4x}{\sin 5x}$

(iii) $\lim_{x\to\infty} \dfrac{x^2 - 4x - 1}{x + 2}$

(iv) $\lim_{x\to 3} \dfrac{x - 3}{x^2 - 2x - 3}$

(v) $\lim_{x\to\infty} \dfrac{1 - 2x + x^3}{2 + 3x - 4x^3}$

(vi) $\lim_{x\to 7} \dfrac{\sqrt{x + 2} - 3}{x - 7}$

(vii) $\lim_{x\to 3} \left(\dfrac{1}{\sqrt{x + 1}} - \dfrac{1}{2} \right)$

(viii) $\lim_{x\to 0} \dfrac{\sin 3x}{\tan 5x}$

(ix) $\lim_{x\to 2^-} \dfrac{x^3 - 5}{x - 2}$

(x) $\lim_{x\to 0} \dfrac{\sin x - x}{x^3}$

(xi) $\lim_{x\to\infty} x^3 e^x$

(xii) $\lim_{x\to 0} \dfrac{e^x - 1}{\sin 2x}$

17. (i) State the Mean Value Theorem, being careful to state the conditions correctly.

(ii) Find the point c for which

$$f(x) = x^2 - 3x + 1$$

satisfies the conclusion of the Mean Value Theorem applied to the interval $[0, 2]$.

(iii) What about the function

$$g(x) = \frac{1}{x}$$

on the interval $[-1, 1]$? Why is it not appropriate to use the Mean Value Theorem in this case?

18. (i) If a function $g(x)$ is continuous on $[a, b]$, differentiable on (a, b), and such that $g(a) = g(b)$, what does Rolle's Theorem tell us?

(ii) For the function

$$f(x) = x^9 + 3x - 3$$

by applying the Intermediate Value Theorem, show that f has a root in the interval $0 < x < 1$.

(iii) Show that f' is always positive.

(iv) Let a in the interval $(0, 1)$ be such that $f(a) = 0$. Suppose that there exists b such that $a \neq b$ and $f(b) = 0$. By applying Rolle's Theorem and considering your answer in (iii), explain why no such b can exist.

(v) Repeat this procedure for the following functions:
(a) $f(x) = 2x^3 + x - 1$
(b) $f(x) = 3x^{11} + 2x - 3$

19. The Mean Value Theorem can often be used in place of Rolle's Theorem. Consider the function

$$g(x) = 2x^5 + 3x - 4.$$

Show, by using the Intermediate Value Theorem, that g has a root in the interval $0 < x < 1$. Now, by using the *Mean Value Theorem*, show that this is the only possible root.

20. The graph of $y^2 = x^3 + x^2$ is given in Figure 1.

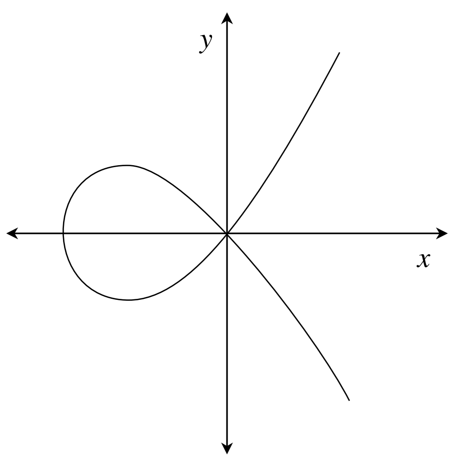

FIGURE 1. The graph of $y^2 = x^3 + x^2$.

(i) When is this graph parallel to the x-axis? Be sure to check that your answer is sensible by comparing it with the graph.

(ii) Have you any comments to make about your result in (i)?

21. Let $f(x) = 3x^4 - 8x^3 + 6x^2 + 1$.

(i) Show that $f'(x) = 12x(x-1)^2$.

(ii) Find all the critical numbers for f, if any.

(iii) Determine the intervals on which f is increasing or decreasing.

(iv) Find all local maxima or minima for f, if any.

(v) Show that $f''(x) = 12(x-1)(3x-1)$.

(vi) Determine the intervals on which f is concave up or concave down.

(vii) Sketch the graph of $y = f(x)$.

22. Let $f(x) = -x^3 + 3x^2 - 1$.

(i) Find the critical numbers for f.

(ii) Determine the intervals on which f is increasing or decreasing.

(iii) Find all local minima and maxima for f.

(iv) Determine the intervals on which f is concave up or concave down.

(v) Find the points of inflection, if any.

(vi) Sketch the graph of $y = f(x)$.

23. Let $f(x) = \dfrac{x^2}{x^2+1}$.

(i) Verify that $f'(x) = \dfrac{2x}{(x^2+1)^2}$, and that $f''(x) = \dfrac{2-6x^2}{(x^2+1)^3}$.

(ii) Find all critical points for f.

(iii) Find all vertical and horizontal asymptotes for the graph of $y = f(x)$.

(iv) Determine all intervals where f is increasing.

(v) Determine all intervals where the graph is concave down.

(vi) Sketch the graph.

24. Let $f(x) = \dfrac{x-1}{x^2}$. You may assume that:

$$f'(x) = \frac{2-x}{x^3} \qquad \text{and that} \qquad f''(x) = \frac{2(x-3)}{x^4}.$$

(i) When is f increasing? When is f decreasing?

(ii) Find all local maxima and minima of f.

(iii) What is the maximum value of f on the closed interval $[1, 2]$?

(iv) When if f concave up? When is f concave down? Are there any points of inflection?

(v) Find the vertical and horizontal asymptotes, if any.

(vi) Sketch the graph of $y = f(x)$.

25. A cylindrical container with no top is to have a volume of 45cm^3. The material for the bottom of the pot costs \$5/cm^2, and for the curved side costs \$3/cm^2. What dimensions will minimise the total cost of this container?

26. A rectangle has an area of 64cm^2. A straight line is to be drawn from one corner of the rectangle to the midpoint of one of the two more distant sides. What is the minimum possible length of such a line?

27. Consider the triangle in the xy-plane whose vertices are at $(0, 0)$, $(0, 1)$, and $(5, 0)$. Find the dimensions of the largest rectangle, with sides parallel to the x and y axis, that can be inscribed in the triangle. Justify your answer using the methods of calculus.

28. The space within a race track of length 2 km is to consist of a rectangle with a semi-circular area at each end. To what dimensions should the track be built to maximise the area of the rectangle? Justify why there is a maximum.