MATH6991

SCHEMES AND GEOMETRY
DECEMBER 2006

This exam consists of four questions, each worth 25 marks. Please attempt all the questions. This is an open–book exam. State clearly any results you use, and if relevant give a reference.

1. Let $\mathbb{A}^n := \operatorname{Spec} k[x_1, \ldots, x_n]$, where k is an algebraically closed field.

(i) State a relevant form of the Nullstellensatz. What are the closed points of $\mathbb{A}^n$? Sketch a diagram representing the points of $\mathbb{A}^1$, making sure to include any generic points.

(ii) Prove that any irreducible polynomial $f(x, y) \in k[x, y]$ is associated with a prime ideal $(f) \subset k[x, y]$ whose closure consists of the point itself and all the closed points (λ, μ) with $f(\lambda, \mu) = 0$. Sketch a diagram representing the points of $\mathbb{A}^2$, making sure to include any generic points.

(iii) Let $U = \mathbb{A}^n \setminus \{0\}$, where $n \geq 2$. By expressing $U = \cup_{i=1}^n X_{x_i}$ as the union of distinguished open subsets, show that $\mathcal{O}_{\mathbb{A}^n}(U) = k[x_1, \ldots, x_n]$. (Hint: We can express $f \in \mathcal{O}_{\mathbb{A}^n}(U)$ as $f = f_i/x_i^{m_i}$ on X_{x_i}, where $f_i \in k[x_1, \ldots, x_n]$.) Hence or otherwise explain why U is not an affine open set.

2. Let $\mathfrak{p}$ be a prime ideal of the ring $\mathbb{Z}[x]$. Suppose that $\mathfrak{p} \cap \mathbb{Z} \neq \{0\}$. Show that $\mathfrak{p} \cap \mathbb{Z} = (p)$ for some prime p. By considering the homomorphism

$$\begin{aligned} \phi : \mathbb{Z}[x] &\to \mathbb{F}_p[x] \\ f(x) &\mapsto f(x) \pmod{p}, \end{aligned}$$

prove that the ideal $\overline{\mathfrak{p}}$ in $\mathbb{F}_p[x]$ generated by $\phi(\mathfrak{p})$ is a prime ideal, and hence can be written in the form $\overline{\mathfrak{p}} = (\overline{h(x)})$ for some irreducible polynomial $\overline{h(x)}$ in $\mathbb{F}_p[x]$. Show that $\mathfrak{p} = (p, f(x))$ for any $f(x) \in \mathbb{Z}[x]$ such that $\phi(f(x)) = \overline{h(x)}$. Hence, or otherwise, find an expression for $V((p))$ (considered as a set).

3. Let $X = \operatorname{Spec} R$ be an affine scheme, with structure sheaf $\mathcal{O}_X$.

(i) State an important result which relates an appropriate localisation of R with sections of $\mathcal{O}_X$ over distinguished open subsets X_f. Hence for any open set U in $\operatorname{Spec} \mathbb{Z}$ find an **explicit** description of $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}}(U)$.

(ii) Find an explicit description of $\mathcal{O}_{\mathbb{A}^1}(U')$ for any open set U' in $\mathbb{A}^1$. Let $g(x) \in k[x]$ be an irreducible polynomial. Let $(g(x))$ be the point in $\mathbb{A}^1$ corresponding to $g(x)$. Find an explicit description of $\mathcal{O}_{\mathbb{A}^1,(g(x))}$.

(iii) Let $\tilde{g}(x)$ be the image of $g(x)$ in $\mathcal{O}_{\mathbb{A}^1,(g(x))}$. Show that $\mathcal{O}_{\mathbb{A}^1,(g(x))}/(\tilde{g}(x))$ is isomorphic to $k[x]/(g(x))$. What does $\mathcal{O}_{\mathbb{A}^1,(0)}$ equal?

4. (i) What does it mean to say that a scheme X is *irreducible*? What does it mean to say that X is *reduced*? Prove that if X is reduced then the stalk $\mathcal{O}_{X,x}$ at x is nilpotent free.

(ii) State, with justification, whether $\operatorname{Spec} k[x,y]/(xy)$ is a reduced scheme. Is $\operatorname{Spec} k[x]/(x^2)$ reduced? Is it irreducible? Give justifications for your answers.

(iii) Let $\phi : \operatorname{Spec} k[x,y]/(xy) \to \mathbb{A}^1$ be the map of affine schemes induced by the ring map:

$$k[t] \to k[x,y]/(xy)$$
$$t \mapsto x + y.$$

Show that the fibre over the point (t) is isomorphic to $\operatorname{Spec} k[x]/(x^2)$.